ME 321: Fluid Mechanics-I

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Lecture - 05 (17/05/2025)
Fluid Dynamics: Continuity Principle

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Recap

Reynolds transport theorem (RTT) for a fixed, nondeforming control volume (CV)

$$\frac{d}{dt}\left(B_{\text{syst}}\right) = \frac{d}{dt}\left(\int_{\text{CV}} \beta\, \rho\, d\forall\right) + \int_{\text{CS}} \beta\, \rho \left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right) dA$$

This relation permits to change from a system approach to control volume (CV) approach.

where

B_{syst} = any property of fluid (mass, momentum, enthalpy, etc.)

β = intensive property of fluid (per unit mass basis)

ρ = density of fluid

$d\forall$ = elemental volume

$\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right) dA$ = elemental volume flux

$\int_{\text{CV}}$ = volume integral over the control volume (CV)

$\int_{\text{CS}}$ = surface integral over the control surface (CS)

Similar expression adopted by other books:

$$\frac{D}{Dt}\left(B_{\text{syst}}\right) = \frac{\partial}{\partial t}\left(\int_{\text{CV}} \beta\, \rho\, d\forall\right) + \int_{\text{CS}} \beta\, \rho \left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right) dA$$

Conservation of Mass

A **system** is defined as a fixed quantity of mass, denoted by *m.* Thus, the mass of the system is conserved and does not change [except nuclear reaction] . so the **conservation of mass principle** for a system is simply stated as

$$m_{\text{syst}} = \text{const.}$$

$$\therefore \frac{dm_{\text{syst}}}{dt} = 0 \qquad (i)$$

Reynolds transport theorem (RTT) with *B* = mass and so, *β* = 1; accordingly

$$\beta = \frac{\text{mass}}{\text{mass}} = 1$$

$$\frac{d}{dt}\left(B_{\text{syst}}\right) = \frac{d}{dt}\left(\int_{CV} \beta\,\rho\,d\forall\right) + \int_{CS} \beta\,\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA$$

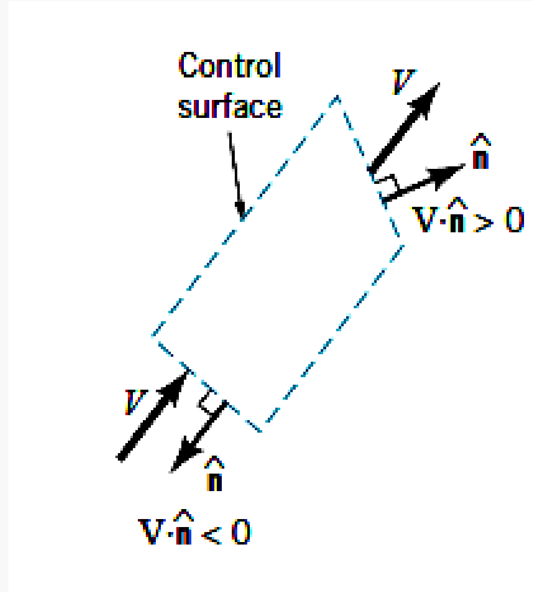
$$\Rightarrow \frac{d}{dt}\left(m_{\text{syst}}\right) = \frac{d}{dt}\left(\int_{CV} \rho\,d\forall\right) + \int_{CS} \rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA$$

$$\Rightarrow \frac{d}{dt}\left(\int_{CV} \rho\,d\forall\right) + \int_{CS} \rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA = 0$$

Control volume expression for conservation of mass, commonly known as **continuity equation**.

Conservation of Mass

For steady flow i.e. $\frac{d}{dt}(\ )=0$



$$\underbrace{\frac{d}{dt}\int_{CV}\rho\, d\forall}_{=0} + \int_{CS}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA = 0$$

$$\Rightarrow \int_{CS}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA = 0 \qquad (ii)$$

The integrand in the mass flow rate integral represents the product of the component of velocity, **V** perpendicular to the small portion of the control surface and the differential area, dA.

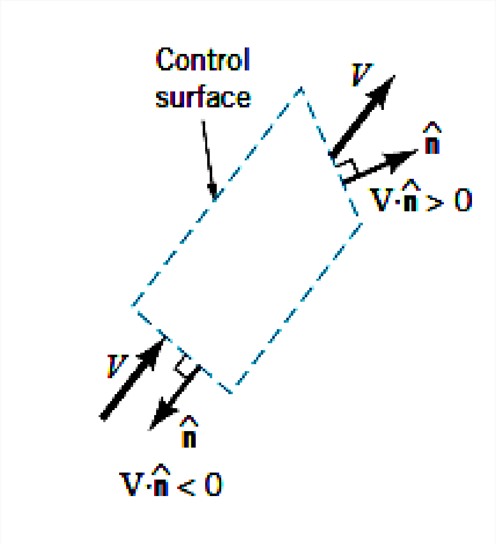
As shown in figure (dot product)

$\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right) = +\text{ve}$; **+ve for flow out** from the control volume

$\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right) = -\text{ve}$; **-ve for flow in** to the control volume

Equation (ii) states that in steady flow, the mass flows entering and leaving the control volume (CV) must balance exactly.

Conservation of Mass



When all of the differential quantities are summed over the entire control surfaces;

$$\int_{CS} \rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA = 0 \quad \equiv \sum(\rho AV)_{\text{out}} - \sum(\rho AV)_{\text{in}}$$

$$= \sum \dot{m}_{\text{out}} - \sum \dot{m}_{\text{in}} \quad = 0$$

$$\boxed{\Rightarrow \sum \dot{m}_{\text{in}} = \sum \dot{m}_{\text{out}}}$$

Mass continuity equation

For incompressible flows, (ρ =constant through the flow system)

$$\Rightarrow \sum(AV)_{\text{in}} = \sum(AV)_{\text{out}}$$

$$\boxed{\Rightarrow \sum Q_{\text{in}} = \sum Q_{\text{out}}}$$

volume continuity equation

Problem

A worker is performing maintenance in a small rectangular tank with a height of 3 m and square base 1.8 m by 1.8 m. Fresh air enters though a 200 mm diameter hose and exists through a 100 mm diameter port on the tank wall. Assume the flow to be steady and incompressible.

(a) Determine the exchange rate needed for the ventilation safety of the worker inside the tank. A complete change of air every 3 minutes (Air Change per Hour, ACH = 20) has been generally accepted by industry as per ventilation requirement.

(b) Determine the velocity of the air entering and existing the tank at this exchange rate.

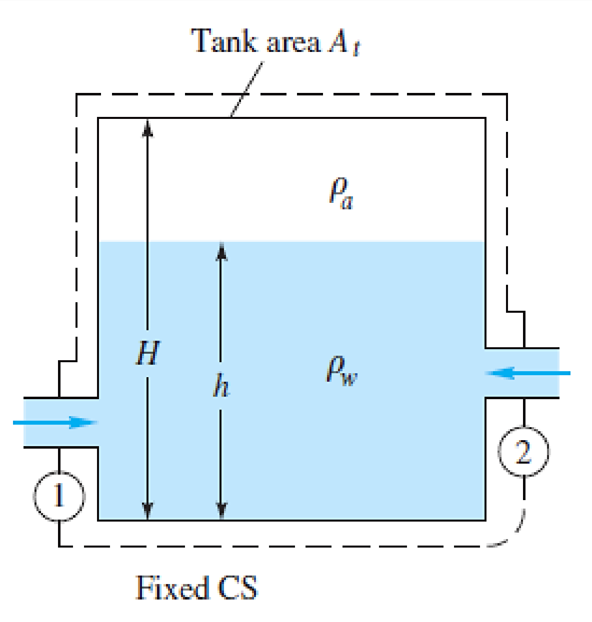
Ans: (a) 3.24 m^3/min (120 cfm)
(b) 1.72 m/s, 6.88 m/s

Problem (Unsteady flow)

The tank in Fig. is being filled with water by two one-dimensional inlets. Air is trapped at the top of the tank. The water height is h.

(a) Find an expression for the change in water height dh/dt.

(b) Compute dh/dt if D_1 = 25 mm, D_2 = 75 mm, V_1 = 0.75 m/s, V_2 = 0.60 m/s, and A_t = 0.2 m^2.



Solution:

General **Continuity Equation** in integral form applied to the shown control volume

$$\frac{d}{dt}\int_{CV}\rho\, d\forall + \int_{CS}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA = 0$$

Unsteady, $\frac{d}{dt}\int_{CV}\rho\, d\forall \neq 0$

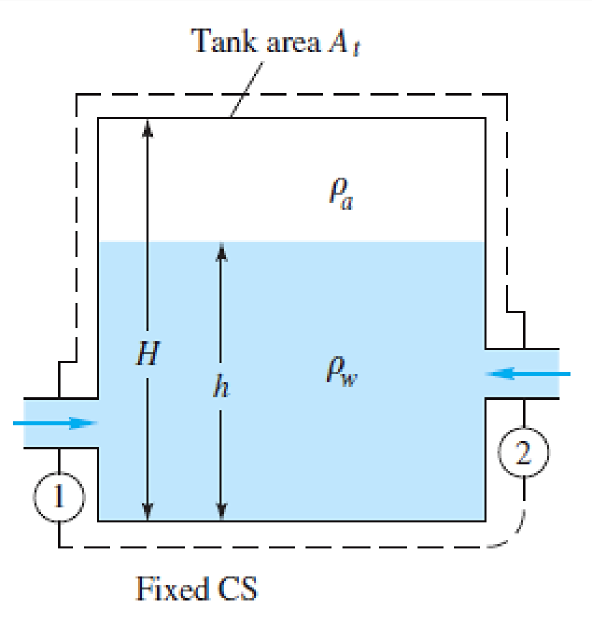
$$\frac{d}{dt}\left(\int_{CV}\rho d\forall\right)-\rho_1 A_1 V_1-\rho_2 A_2 V_2=0$$

Now,

$$\frac{d}{dt}\left(\int_{CV}\rho d\forall\right)=\frac{d}{dt}(m_{CV})=\frac{d}{dt}(\rho_w A_t h)+\frac{d}{dt}[\rho_a A_t(H-h)]$$

= 0

(air is trapped, no change of air mass with time)



$$\Rightarrow \frac{d}{dt}\left(\int_{CV}\rho d\forall\right)=\rho_w A_t\frac{dh}{dt}$$

Thus,

$$\frac{dh}{dt}=\frac{\rho_1 A_1 V_1+\rho_2 A_2 V_2}{\rho_w A_t}$$

$$\Rightarrow \frac{dh}{dt}=\frac{A_1 V_1+A_2 V}{A_t}$$

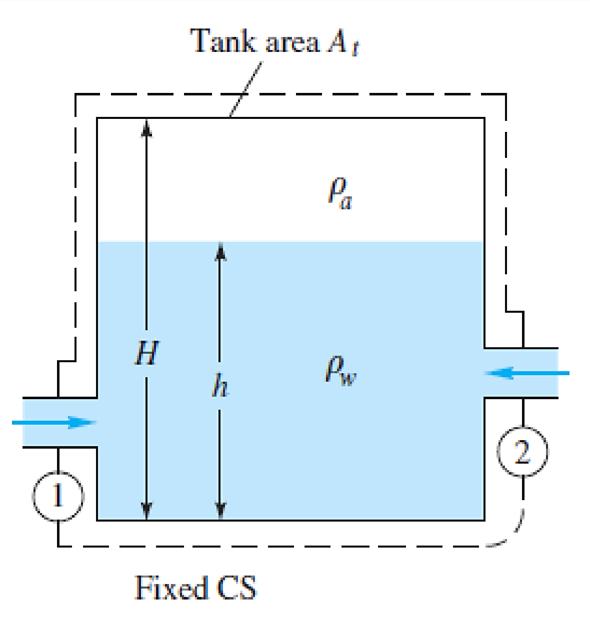
$$\Rightarrow \frac{dh}{dt}=\frac{Q_1+Q_2}{A_t}$$

Ans. (a)

$$\Rightarrow \frac{dh}{dt} = \frac{Q_1 + Q_2}{A_t}$$

$$\Rightarrow \frac{dh}{dt} = \frac{\pi/4 D_1^2 V_1 + \pi/4 D_2^2 V_2}{A_t}$$

$$\Rightarrow \frac{dh}{dt} = 0.015 \text{ m/s}$$ Ans. (b)



Problem (Unsteady flow)

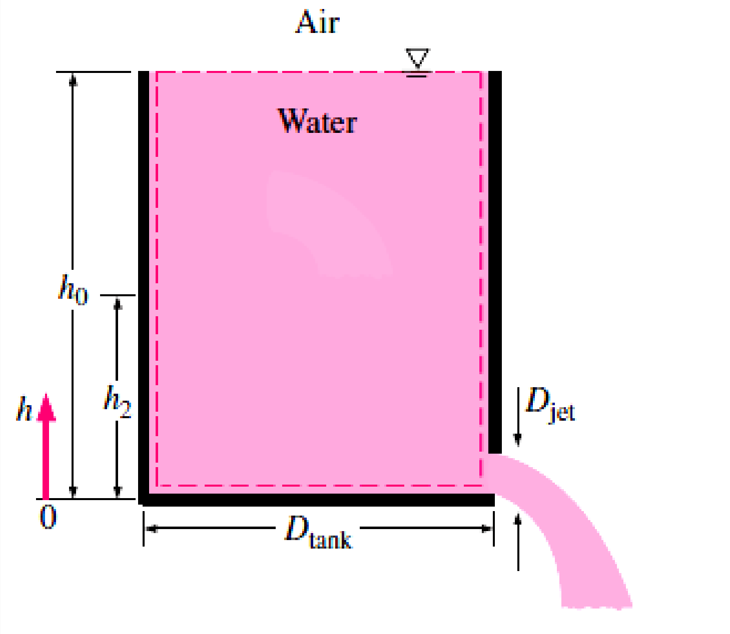
A 1.5 m high, 1 m diameter cylindrical water tank whose top is open to the atmosphere is initially filled with water. Now, the discharge plug near the bottom of the tank is pulled out, and a water jet whose diameter is 0.01 m streams out (Fig.). The average velocity of the jet is given by:

$$V_{jet} = \sqrt{2gh} \qquad (\text{m/s})$$

where *h* is the height of water in the tank measured from the center of the hole and g is the gravitational acceleration. Determine

(i) How long it will take for the water level in the tank to drop to 0.75 m from the bottom?

(ii) How long it will take to empty the tank?



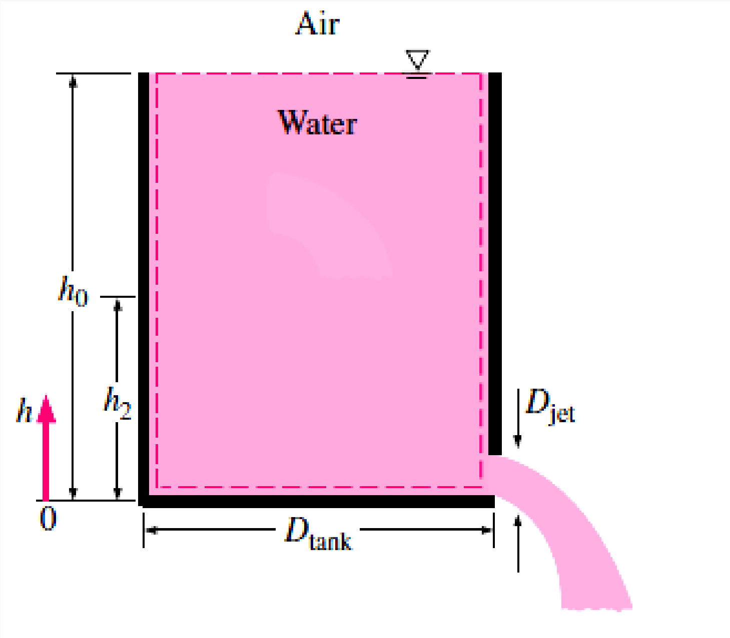
Solution:

General **Continuity Equation** in integral form applied to the shown control volume

$$\frac{d}{dt}\int_{CV} \rho \, d\forall + \int_{CS} \rho \left(\vec{\mathbf{V}} \cdot \hat{\mathbf{n}}\right) dA = 0$$

Unsteady, $\frac{d}{dt}\int_{CV} \rho \, d\forall \neq 0$

Problem (Unsteady flow)



Now, $$\frac{d}{dt}\int_{CV}\rho\, d\forall + \int_{CS}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA = 0$$

$$\Rightarrow \frac{d}{dt}\left(m_{CV}\right) + \rho A_{jet}V_{jet} = 0$$

No inflow; only one outflow through the hole (+ve)

$$m_{CV} = \rho\forall = \rho\left(\frac{\pi}{4}D_{tank}^2 \times h\right)$$

$$h = h(t);\quad m_{CV} = f(t)$$

$$\rho A_{jet}V_{jet} = \rho\left(\frac{\pi}{4}D_{jet}^2\right)\sqrt{2gh}$$

$$V_{jet} = \sqrt{2gh} \quad = f(t)$$

Then,

$$\Rightarrow \frac{d}{dt}\left\{\rho\left(\frac{\pi}{4}D_{tank}^2 \times h\right)\right\} + \rho\left(\frac{\pi}{4}D_{jet}^2\right)\sqrt{2gh} = 0$$

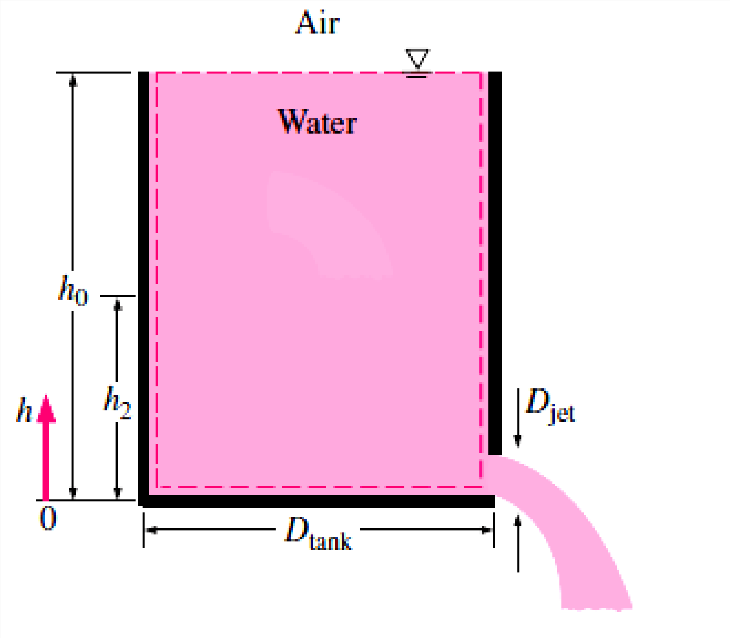
$$\Rightarrow \frac{d}{dt}\left\{\left(D_{tank}^2 \times h\right)\right\} = -\left(D_{jet}^2\right)\sqrt{2gh}$$

$$\Rightarrow \frac{dh}{dt} = -\left(\frac{D_{jet}^2}{D_{tank}^2}\right)\sqrt{2gh}$$

Unsteady, $\frac{\partial}{\partial t}\int_{CV}\rho\, d\forall \neq 0$

Problem (Unsteady flow)

$$\Rightarrow dt = -\frac{1}{\sqrt{2g}}\frac{D_{\text{tank}}^2}{D_{\text{jet}}^2}\frac{dh}{\sqrt{h}}$$



Now, integrating from t = 0 at which $h = h_0$ to $t = t$ at which $h = h_t$

$$\int_0^t dt = -\frac{1}{\sqrt{2g}}\frac{D_{\text{tank}}^2}{D_{\text{jet}}^2}\int_{h_0}^{h_t}\frac{dh}{\sqrt{h}}$$

$$\Rightarrow t = -\frac{1}{\sqrt{2g}}\frac{D_{\text{tank}}^2}{D_{\text{jet}}^2}\left[\frac{h^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]_{h_0}^{h_t}$$

$$\Rightarrow t = -\frac{1}{\sqrt{g/2}}\frac{D_{\text{tank}}^2}{D_{\text{jet}}^2}\left|\sqrt{h}\right|_{h_0}^{h_t}$$

$$\Rightarrow t = \frac{\sqrt{h_0}-\sqrt{h_t}}{\sqrt{g/2}}\frac{D_{\text{tank}}^2}{D_{\text{jet}}^2}$$

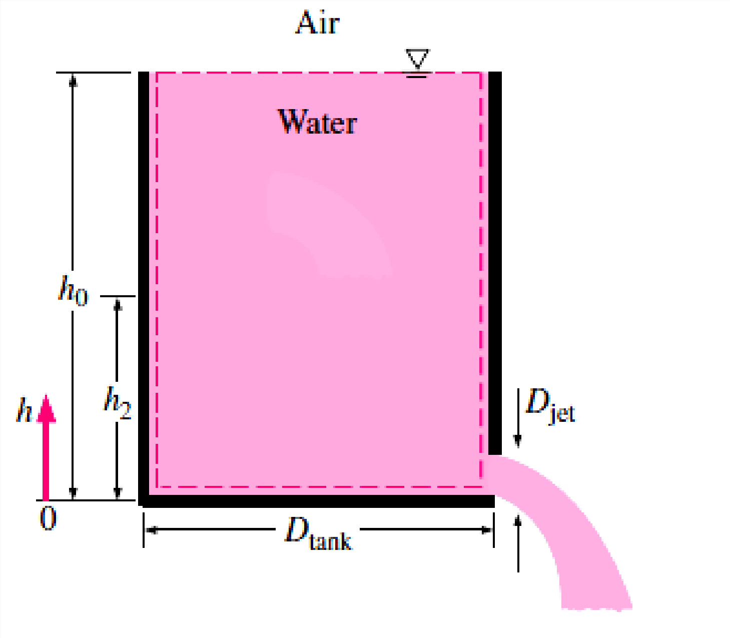
Time required to reduce the water height from h_0 to h_t

Problem (Unsteady flow)

Time required for the water level in the tank to drop to 0.75 m from the bottom:

$$t = \frac{\sqrt{h_0} - \sqrt{h_t}}{\sqrt{g/2}} \frac{D_{\text{tank}}^2}{D_{\text{jet}}^2}$$

$$\therefore t_{h_t=0.75} = \frac{\sqrt{1.5} - \sqrt{0.75}}{\sqrt{g/2}} \frac{1^2}{0.01^2} = 1619.7\,\text{s} = 27\,\text{min}$$



Time required to empty the water tank:

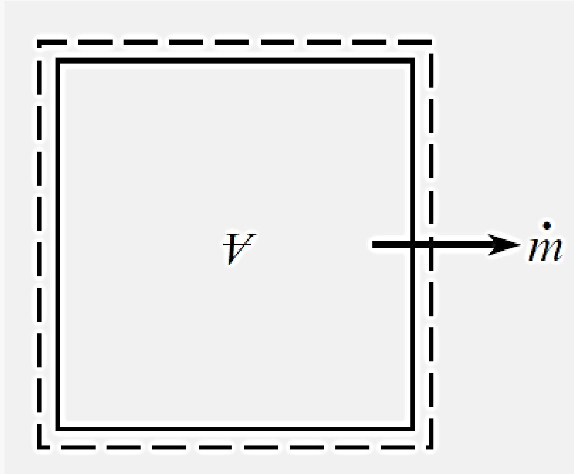
$$\therefore t_{h_t=0} = \frac{\sqrt{1.5} - \sqrt{0}}{\sqrt{g/2}} \frac{1^2}{0.01^2} = 5530\,\text{s} = 92\,\text{min}$$

Time requirement is **NOT linear (rather non-linear) (AN UNSTEADY PROBLEM)**

Homework:
Plot the water height, *h* versus time, *t*

Problem (Unsteady flow)

Methane escapes through a small (10^{-7} m^2) hole in a 10 m^3 tank. The methane escapes so slowly that the temperature in the tank remains constant at 23°C. The mass flow rate of methane through the hole is given by $\dot{m} = 0.66\, pA/\sqrt{RT}$, where p is the pressure in the tank, A is the area of the hole, R is the gas constant, and T is the temperature in the tank. Calculate the time required for the absolute pressure in the tank to decrease from 500 to 400 kPa.



- There is no mass inflow:

$$\sum_{cs} \dot{m}_i = 0$$

- Mass out flow rate is

$$\sum_{cs} \dot{m}_o = 0.66 \frac{pA}{\sqrt{RT}}$$

Substituting terms into the continuity equation gives

$$Ɐ \frac{d\rho}{dt} = -0.66 \frac{pA}{\sqrt{RT}}$$

3. Equation for elapsed time:
- Use ideal gas law for ρ:

$$Ɐ \frac{d}{dt}\left(\frac{p}{RT}\right) = -0.66 \frac{pA}{\sqrt{RT}}$$

- Because R and T are constant,

$$\frac{dp}{dt} = -0.66 \frac{pA\sqrt{RT}}{Ɐ}$$

- Next, separate variables:

$$\frac{dp}{p} = -0.66 \frac{A\sqrt{RT}\,dt}{Ɐ}$$

- Integrating the equation and substituting limits for initial and final pressure gives

$$t = \frac{1.52Ɐ}{A\sqrt{RT}} \ln \frac{p_0}{p_f}$$

4. Elapsed time:

$$t = \frac{1.52\,(10\ \text{m}^3)}{(10^{-7}\ \text{m}^2)\left(518 \frac{\text{J}}{\text{kg}\cdot\text{K}} \times 300\ \text{K}\right)^{1/2}} \ln \frac{500}{400} = \boxed{8.6 \times 10^4\ \text{s}}$$

Review the Solution and the Process

1. *Discussion.* The time corresponds to approximately one day.
2. *Knowledge.* Because the ideal gas law is used, the pressure and temperature have to be in absolute values.

Problem

Consider a highly pressurized air tank at conditions (p_0, ρ_0, T_0) and volume V_0. It is known from compressible flow theory, that if the tank is allowed to exhaust to the atmosphere through a well-designed converging nozzle of exit area A, the outgoing mass flow rate will be

$$\dot{m} = \frac{\alpha p_0 A}{\sqrt{RT_0}} \quad \text{, where } \alpha \approx 0.685 \text{ for air}$$

This rate persists as long as p_0 is at least twice as large as the atmospheric pressure. Assuming constant T_0 and ideal gas,

(a) Derive a formula for the change of density $\rho_0(t)$ within the tank.

(b) Estimate the time required for the density to decrease by 25%.

Solution:

$$\dot{m} = \frac{\alpha\, p_o\, A}{\sqrt{RT_o}} = \frac{\alpha\,(\rho_o RT_o)\,A}{\sqrt{RT_o}} = \alpha\,\rho_o\,A\sqrt{RT_o}$$

Now apply a mass balance to a control volume surrounding the tank:

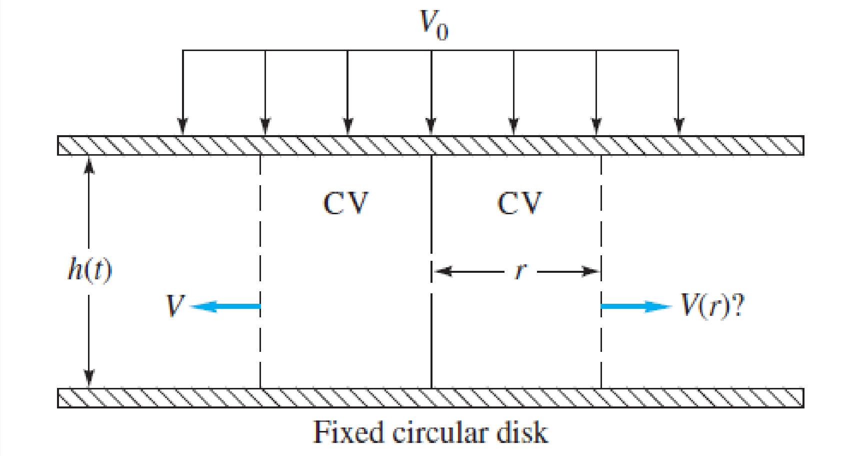
$$\frac{dm}{dt}\Big|_{system} = 0 = \frac{d}{dt}(\rho_o \upsilon_o) + \dot{m}_{out} = \upsilon_o \frac{d\rho_o}{dt} + \alpha\rho_o A\sqrt{RT_o}$$

$$\frac{\rho_{o2}}{\rho_{o1}} = \exp\left[\frac{-\alpha A\sqrt{RT_o}}{\upsilon_o}(t_2 - t_1)\right] \quad Ans.(a)$$

$$\Delta t = \frac{0.288\,\upsilon_o}{\alpha A\sqrt{RT_o}} \quad Ans.(b)$$

Problem

An incompressible fluid is being squeezed outward between two large circular disks by the uniform downward motion V_0 of the upper disk. Assuming one dimensional radial outflow, use the control volume shown to derive an expression for $V(r)$.



Solution: General Continuity Equation in integral form applied to the shown control volume:

$$\frac{d}{dt}\int_{CV} \rho \, d\forall + \int_{CS} \rho(\vec{\mathbf{V}} \cdot \hat{\mathbf{n}}) dA = 0$$

$$\frac{d}{dt}(\rho\pi r^2 h(t)) + \rho(2\pi r h(t))V(r) = 0$$

$$r\frac{dh}{dt} + (2h)V(r) = 0$$

$$r(-V_0) + (2h)V(r) = 0 \qquad ; \frac{dh}{dt} = -V_0 \quad \text{; Upper disk velocity} \qquad h(t) = h_0 - V_0 t$$

$$V(r) = V_0\left(\frac{r}{2h}\right)$$